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POP approximation to the spectral dimension of dual three-dimensional Sierpinski carpets

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Abstract. The spectral dimension $\tilde{d}$ of a network governs the massless singularity of a free field and the asymptotic behaviour of the diffusion on the network. Approximate values of $\tilde{d}$ for two types of three-dimensional generalisations of the dual Sierpinski carpet are obtained using the POP approximation method. For one of them which is generated by a cube of side length three, with one block at the centre taken away, the value obtained is $\tilde{d}(\text{POP}) = 2\log 26/\log(884/93) \approx 2.89$. For the other one with seven cubes taken away, $\tilde{d}(\text{POP}) = 2\log 20/\log(40/3) \approx 2.31$. The algorithm of the POP method is explained. The results for 2D symmetric dual Sierpinski-type carpets are also reported.

1. Introduction

This paper concerns the spectral dimension $\tilde{d}$ [1] of a network defined through the massless singularity of a free (i.e. Gaussian) field on the network [2-5]:

$$\langle \phi_x^2 \rangle (J, g) \sim \text{constant} \times g^{(\tilde{d}-2)/2} \qquad \text{as } g \downarrow 0.$$

Here $J = \{J_{xy}\}_{x,y \in V}$ are the interactions between the spins on different sites, $g = \{g_x\}_{x \in V}$ are the parameters (square of mass) that appear in the single-site measure:

$$\sim \exp(-g\phi^2/2)\, \mathrm{d}\phi.$$

With this definition, the spectral dimension is essentially equivalent to the exponent that governs the long-time asymptotic behaviour of the diffusion (random walk) on the network [5, 6]:

$$u_0(\tau) \sim \tau^{-\tilde{d}/2} \qquad \tau \to \infty.$$

For regular translationally invariant lattices, $\tilde{d}$ is equal to the usual dimension, the number of generators of translations. This result is obtained by the Fourier transformation method.

In general, for systems without translational invariance, $\tilde{d}$ may take non-integer values. Except for a limited type of system, it is not easy to obtain this value. There are examples where non-integer values of $\tilde{d}$ are obtained exactly [5-10], but the method (coarse-graining renormalisation) used to obtain them cannot be applied to the networks with $\tilde{d} > 2$ [3]. Furthermore, the coarse-graining renormalisation method is effective only for finitely ramified fractals. For infinitely ramified fractals such as the Sierpinski carpet, the exact value of $\tilde{d}$ is not known.

We proposed in our previous work [2] a new approximation method (POP approximation). This approximation was introduced to give an approximate value of the spectral dimension for the Sierpinski carpet. It was inspired by the average-value block spin renormalisation group method. Though at the time when we proposed the method we had results only for networks with $\tilde{d} < 2$, it is an approximation method which is available also for networks with $\tilde{d} > 2$. As examples of networks with $\tilde{d} > 2$, we report our results for two kinds of generalisations of the original Sierpinski carpet to three dimensions. Our results are

$$\tilde{d}(\text{POP; SC3a}^*) = 2\log 26/\log(884/93) \approx 2.89$$

$$\tilde{d}(\text{POP; SC3b}^*) = 2\log 20/\log(40/3) \approx 2.31.$$

(Notations are described in § 2.2.) The exact values of the spectral dimensions of the Sierpinski carpet and its generalisations are not known.

We also report the algorithm of POP approximations, and the results for symmetric generalisations of two-dimensional Sierpinski carpets.

In § 2, we formulate free field theory on a network and give our definition of the spectral dimension. We also introduce two kinds of generalisation of the original Sierpinski carpet to three dimensions, and a group of two-dimensional generalised symmetric Sierpinski-type carpets. In § 3, we state the algorithm of the POP approximation. In § 4, we present our results, the approximate values of the spectral dimensions, obtained by applying POP approximation to free fields on the dual three-dimensional Sierpinski carpets and the dual two-dimensional symmetric Sierpinski-type carpets. Then we discuss what we could observe from our results. In § 5, we give conjectures and discussions on the validity of our approximations.

2. Definitions

2.1. Free field theory on a network and the spectral dimension

Let V be a set of vertices (sites). We denote each site as $x, y, \ldots$. We join the sites by a certain set of links (bonds) B, and obtain a network (V, B). In the next subsection we will give the examples of networks which we consider in the present work. We introduce a free field theory (Gaussian spin system) on the network (V, B). The probability measure (statistical weight) of a field configuration $\phi = (\phi_x)_{x \in V}$ is given by

$$\mathrm{d}\mu(\phi) = Z^{-1} \exp\{-S(\phi)\} \prod_{x \in V} \mathrm{d}\phi_x$$

where the action S is

$$S = \frac{1}{2}\left(\frac{1}{2}\sum_{x,y \in V} J_{xy}(\phi_x - \phi_y)^2 + \sum_{x \in V} g_x \phi_x^2\right). \tag{1}$$

Here J_{xy} $(x, y \in V)$ are non-negative constants, which we shall call 'interactions', and g_x $(x \in V)$ are positive constants ('square of mass'). We regard this theory as a field theory on the network (V, B) by imposing

$$J_{xy} = 0 \qquad \text{if } (x, y) \notin B$$
$$\neq 0 \qquad \text{if } (x, y) \in B.$$

(Strictly speaking, (1) is a formal expression and we should, for example, start with the finite volume version and then take the infinite volume limit. See remark 1 at the end of this subsection.)

In general, J_{xy} and g_x may take different values for different bonds and sites (see [4, 5] for the effect of J and g taking different values), but in this work we shall consider the cases where they are constants. Thus we may write

$$S=\frac{1}{2}\left(J\sum_{(x,y)\in B}(\phi_x-\phi_y)^2+g\sum_{x\in V}\phi_x^2\right). \tag{1'}$$

If we consider the points labelled by V and join points x and y by a line if $(x, y)\in B$, we obtain a figure which is in one-to-one correspondence with the field theory that we are considering. The figures that we give in this paper represent the field theories in this way.

The expectation value (correlation function) of a functional $F(\phi)$ in field configuration ϕ is given, as usual, by

$$\langle F(\phi)\rangle=\int F(\phi)\,\mathrm{d}\mu(\phi).$$

In particular, the two-point functions (covariance) are given by

$$\langle\phi_x\phi_y\rangle=\langle\phi_x\phi_y\rangle(J, g)=(M^{-1})_{xy}$$

where the matrix M is given by

$$\begin{aligned}M_{xy}&=\left(g_x+\sum_{z\in V}J_{xz}\right)\delta_{x,y}-J_{xy}\\&=(g+J\times\#\{\text{bonds emerging from }x\})\delta_{x,y}-J_{xy}.\end{aligned}$$

A two-point function $\langle\phi_x\phi_y\rangle(J, g)$ has a non-analyticity (massless singularity) at $g=0$. The spectral dimension $\tilde{d}(V, B)$ of a network (V, B) is an exponent that characterises the massless singularity of the free field on the network:

$$\begin{aligned}\langle\phi_x^2\rangle(J, g)&=\text{constant}\times g^{(\tilde{d}-2)/2}+(\text{regular terms})\\&\quad+(\text{subleading singularities})\qquad \text{as } g\downarrow 0.\end{aligned} \tag{2}$$

The spectral dimension defined in this way also characterises the asymptotic (long-time) behaviour of the diffusion (continuous time random walk) on the network. Consider a random walk on the network (V, B) starting from $x\in V$. The probability $p_y(\tau)$ of finding the random walker on a site y at time τ (>0) obeys the diffusion equations: for $y\in V$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}p_y(\tau)=\sum_{z\in V}J_{yz}(p_z(\tau)-p_y(\tau))$$

$$p_y(0)=\delta_{yx}.$$

Then we obtain

$$p_x(\tau)\sim\tau^{-\tilde{d}/2}\qquad \tau\to\infty.$$

See [5; appendix] for the derivation and the precise statement.

For regular translationally invariant lattices, we obtain $\tilde{d}=d$, where d is the 'canonical' dimension, the number of generators of translations. Therefore we may regard our definition of the spectral dimension as a natural extension of the standard notion of dimension.

The results for regular lattices are direct consequences of the translational invariance. Because of the invariance, one can perform Fourier transformation, and because there are d independent directions, there are d components in the momentum vector, and $\tilde{d}=d$ follows. In fact, let $V=\mathbb{Z}^d$, the d-dimensional (hyper-)cubic lattices, where d is a positive integer (i.e. the vertices are labelled by d component vectors). Let the set of bonds B be the set of nearest-neighbour pairs:

$$B=\{(x,y)|\,|x-y|=1\}$$

and let $J=1$. By Fourier transformation

$$\begin{aligned}\langle\phi_x^2\rangle(J,g)&=(2\pi)^{-d}\int_0^{2\pi}\Bigl(2\sum_{1\leqslant\mu\leqslant d}(1-\cos p_\mu)+g\Bigr)^{-1}\mathrm{d}^d p\\&=(2\pi)^{-d}\int\frac{1}{p^2+g}\mathrm{d}^d p+(\text{regular terms})+(\text{subleading singularities})\\&=\text{constant}\times\begin{cases}g^{(d-2)/2}\log g & (\text{if } d \text{ is even})\\ g^{(d-2)/2} & (\text{if } d \text{ is odd})\end{cases}\\&\qquad+(\text{regular terms})+(\text{subleading singularities})\end{aligned}$$

from which $\tilde{d}=d$ follows.

For systems without translational invariance, $\tilde{d}$ may take non-integer values. Except for a limited type of system, it is quite non-trivial to obtain this value. There are examples where non-integer values of $\tilde{d}$ are obtained exactly, but the method (coarse-graining renormalisation) used to obtain them cannot be used for the networks with $\tilde{d}>2$ [3]. Furthermore, the coarse-graining renormalisation method is, at present, effective only for a special class of finitely ramified fractals [5-10].

Remark 1. Many of the notations introduced here can be made rigorous. However, as our aim in this paper is on an approximation method, we do not give a rigorous formulation here. For rigorous set-ups, see [5].

Remark 2. The existence of spectral dimension for a given network is not trivial [5], but we will not go into this problem. In the present work we will always assume the existence of the spectral dimensions for the networks that we consider.

Remark 3. In case the value of the spectral dimension takes an even integer value for a given network, the first term in (2) is not singular. In this case we read (2) as

$$\langle\phi_x^2\rangle(J,g)=\text{constant}\times g^{(\tilde{d}-2)/2}\log g+(\text{regular terms})+(\text{subleading singularities})\qquad \text{as } g\downarrow 0.$$

Actually, we have already introduced this 'log correction', when we considered the regular lattices.

Remark 4 . We can prove various invariance properties of the spectral dimension $\tilde{d}$ [5; § 3]. (i) $\tilde{d}$ formally depends on the choice of the vertex $x\in V$ that appears in the definition (2), but actually, it is independent. (ii) Under certain conditions, $\tilde{d}$ does not change, even if we vary g_x from site to site. (iii) Again, under certain conditions,

$\tilde{d}$ does not depend on the details of J. We can prove the following type of statement. If there exist positive constants C and C' which are independent of $x \in V$ such that

$$CJ_x < J'_x < C'J_x \qquad x \in V$$

then

$$\tilde{d}(J') = \tilde{d}(J).$$

2.2. *Definition of the three-dimensional Sierpinski carpets and two-dimensional symmetric Sierpinski-type carpets*

We introduce the fractal networks, generalised Sierpinski carpets, which we consider in the following.

To construct a fractal network [11], we start with a basic figure F_0 called an 'initiator'. A generator, or a block, F_1, is constructed by putting several F_0 together in a definite manner. Inductively, F_{n+1} is constructed from F_n in just the same manner as F_1 is constructed from F_0. The limit $F = \lim_{n\to\infty} F_n$ is the fractal network. The first three steps of the construction of the Sierpinski carpet is shown in figure 1.

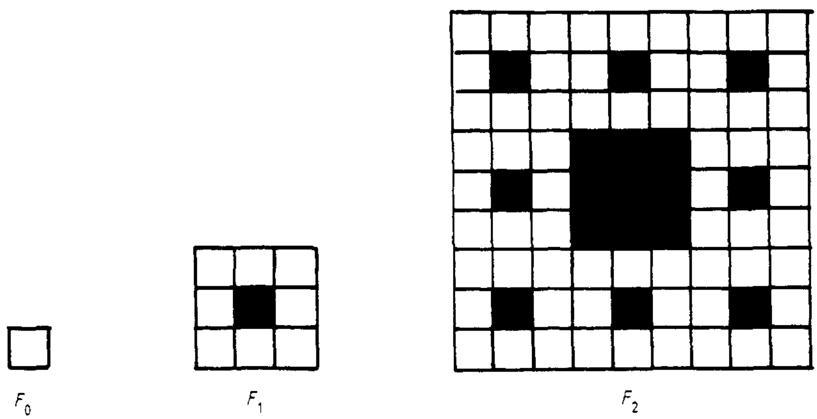

Figure 1. The first three steps of the construction of the Sierpinski carpet.

We consider two kinds of three-dimensional analogue (SC3) of the Sierpinski carpet. (By 'three dimensional' we mean that the network is naturally embedded in a three-dimensional regular lattice. It does *not* mean that $\tilde{d} = 3$.) In either case, the initiator is a cube of size one. Assemble 27 cubes to form a cube of side of length three. Taking away the cube located at the centre, we obtain a generator (block) for a three-dimensional version of the Sierpinski carpet, which we call SC3a. The generator for SC3b, the other version of the three-dimensional Sierpinski carpet, is obtained by taking away seven cubes in such a way that the resulting figure is symmetric with respect to the exchange of the faces (figure 2).

We also consider various types of two-dimensional Sierpinski-type carpets that are symmetric with respect to the interchange in the coordinate axis. We fix the block size to 3×3. Assemble nine squares to form a square of side of length 3. Take some unit squares away in such a way that the resulting figure is symmetric with respect to the interchange in the coordinate axis. Each figure obtained in this way can be a generator

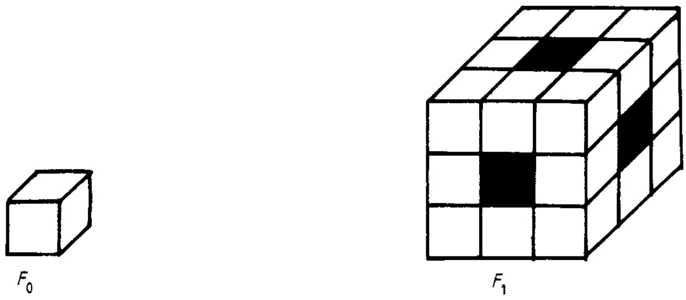

Figure 2. The initiator F_0 and the generator F_1 for SC3b.

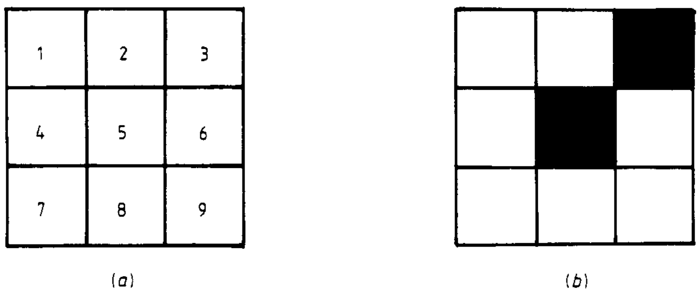

Figure 3. (*a*) The figure that determines our notation for two-dimensional Sierpinski-type carpets (SC2). (*b*) An example of SC2. The generator F_1 for SC2{35}.

(block) of a fractal network and we call this family of networks two-dimensional symmetric Sierpinski-type carpets (SC2). Let us number the unit squares as in figure 3(*a*). Each member of SC2 is specified by which squares to take away. We denote the fractal by SC2$\{n_1 \ldots n_h\}$, if we take away squares $n_1, \ldots, n_h$ (figure 3(*b*)). For example, the original Sierpinski carpet is denoted by SC2{5}.

When we consider fields on the generalised Sierpinski-type carpets, we will consider the field theories on dual Sierpinski-type carpets; for SC2 (SC3), we put the field variables at the centre of each unit square (cube). The bonds are assumed for each pair of nearest-neighbour field variables. We do so because it is the dual carpets, and not the carpets themselves, that are self-similar by the POP renormalisation (scale transformation) procedure. We denote the dual Sierpinski carpets by SC*. By the argument given in [2], we can prove for two-dimensional Sierpinski-type carpets:

$$\tilde{d}(\text{SC2}\{h\}^*) = \tilde{d}(\text{SC2}\{h\}).$$

Our approximate results for two-dimensional dual carpets (which we will give in § 4) therefore serve as approximations for the carpets themselves. We do not have proofs for the case $\tilde{d} > 2$, but we conjecture that similar relations will hold [5].

3. Method of POP approximation

As we have discussed, it is not easy to evaluate the spectral dimension of Sierpinski-type carpets. In [2] an approximation method (POP approximation) was proposed to give an approximate value of $\tilde{d}$ for the (original) Sierpinski carpet. This method was inspired by the average-value block spin renormalisation group method, combined with the Fourier transformation technique. The method gives $\tilde{d}$ through the scaling arguments. For the motivation and the idea of the POP approximation, see [2].

To explain how the scaling arguments enter into our methods, we recall the case of obtaining $\tilde{d}$ for the Sierpinski gaskets using the coarse-graining renormalisation method [5]. There, the coarse graining gives an equality of the form:

$$\langle\phi_x^2\rangle(J, g) = \langle\phi_x^2\rangle(\alpha J'(J, g), \beta g'(J, g)) \tag{3}$$

where α and β are positive constants such that $\beta > \alpha$, $\alpha \neq 1$, and the functions J' and g' obey

$$J'(J, g) \to J \qquad g'(J, g)/g \to 1 \qquad \text{as } g \to 0.$$

The rigorous scaling argument then gives

$$\tilde{d} = \frac{2 \log \beta}{\log(\beta/\alpha)}. \tag{4}$$

The POP approximation is a method that gives an approximate relation of the form given by (3). We then rely on the scaling arguments and use (4) to obtain the spectral dimension. We also gave a heuristic scaling argument for the average-value block-spin renormalisation in [2; appendix], which results in the same conclusion, equation (4). (We note that we followed [3-5] and slightly changed our notation from that of [2]. $\tilde{d}$ and α in [2] correspond to $\log \beta/\log L$ and $\log \alpha/2 \log L$ in this paper, respectively. Here, L is a side length of a block, which is three for the Sierpinski carpet.)

The explicit formula for POP approximation is reached through the following steps (i)-(v). The formula is summarised in (9), (10) and (11). For definiteness, we shall explain the method for the dual Sierpinski-type carpets. The applications to other networks are straightforward. In fact, the POP approximation is applicable to any network that has a natural embedding into a regular lattice.

(i) Choose a regular (translationally invariant) lattice in which the fractal that we are considering is naturally embedded. For SC2* we choose a square lattice $\mathbb{Z}^2$, for SC3* we choose $\mathbb{Z}^3$.

(ii) Introduce a free field theory on the lattice. This means that we choose B in the definition of the action S (equation (1')) to be a set of nearest-neighbour pairs on the lattice. Using a Fourier transformation, we can express S for the regular lattice as

$$S \equiv \tfrac{1}{2} \sum_{x, y} \phi_x M_{xy} \phi_y \tag{5}$$

$$M_{xy} = (2\pi)^{-d} \int_0^{2\pi} \tilde{M}(p) \exp[-\mathrm{i}p \cdot (x-y)] \, \mathrm{d}^d p$$

$$\tilde{M}(p) = g + 2J \sum_{\mu=1}^{d} [1 - \cos(p_\mu)] \tag{6}$$

where $d = 2$ for SC2*.

(iii) Divide the regular lattice into blocks (3×3 sublattices for SC2*). We arrange it so that the origin of the coordinate system falls just on the centre of one of the blocks. Remove the field variables and the interactions in the action S of equation (5) in such a way that each block has the same pattern as the generator (block) of the fractal that we are considering. Notice that the resulting network $F = (V^1, B^1)$ still has translational invariance in the unit of three lattice spacing, though it is not invariant in the translation of one lattice spacing. (This procedure corresponds to the operation P_1 in the notation of [2].) We introduce a new coordinate $\tilde{x}$ which is three times larger in scale than the original coordinate: i.e. $3\tilde{x}$ indicates the (original) coordinate of the centre of each block. We also rename each field variable ϕ_x as $\phi_{\tilde{y},j}$, if x is a site in a block indicated by $\tilde{y}$. Index j is introduced to discriminate the sites in a block. We can rewrite the resulting action as:

$$P_1(S)=\tfrac{1}{2}\sum_{\tilde{x},\tilde{y}} {}^t\boldsymbol{\phi}_{\tilde{x}} M'_{\tilde{x}\tilde{y}} \boldsymbol{\phi}_{\tilde{y}}.$$

Here $\boldsymbol{\phi}_{\tilde{x}}$ is a column vector which is composed of the field variables ϕ_x in the block indicated by $\tilde{x}$:

$$\boldsymbol{\phi}_{\tilde{x}}=\begin{bmatrix}\phi_{\tilde{x},1}\\ \vdots \\ \phi_{\tilde{x},n}\end{bmatrix}$$

where n is the number of the field variables (degree of freedom) in a block, and $M'_{\tilde{x}\tilde{y}}$ is an $n\times n$ matrix. Applying Fourier transformation to the coordinate $\tilde{x}$ we can write it in the form:

$$M'_{\tilde{x}\tilde{y}}=(2\pi)^{-d}\int_0^{2\pi} C(p)^{-1}\exp[-\mathrm{i}p\cdot(\tilde{x}-\tilde{y})]\,\mathrm{d}^d p$$

$$C(p)=A+\sum_{\mu=1}^{d}[B_\mu \exp(-\mathrm{i}p_\mu)+{}^tB_\mu \exp(\mathrm{i}p_\mu)]. \tag{7}$$

A and B_μ are $n\times n$ matrices, whose entries are given by J and g.

(iv) Introduce the block spin field $\phi^1_{\tilde{x}}$ that lives on $\tilde{x}$, by:

$$\phi^1_{\tilde{x}}\equiv(O\phi)_{\tilde{x}}=\frac{1}{n}\sum_{x'\in b_{\tilde{x}}}\phi_{x'}$$

where $b_{\tilde{x}}$ is a block in F indicated by $\tilde{x}$. It is an elementary fact about block-spin operations for Gaussian fields that the effective action for the block spin field becomes

$$OP_1(S)(\phi^1)=\tfrac{1}{2}\sum_{\tilde{x},\tilde{y}}\phi^1_{\tilde{x}}M^1_{\tilde{x}\tilde{y}}\phi^1_{\tilde{y}}$$

where

$$\{(M^1)^{-1}\}_{\tilde{x}\tilde{y}}=(OM'^{-1}{}^tO)_{\tilde{x}\tilde{y}}$$
$$=(2\pi)^{-d}\int_0^{2\pi}\tilde{M}^1(p)^{-1}\exp[-\mathrm{i}p\cdot(\tilde{x}-\tilde{y})]\,\mathrm{d}^d p$$

$$\tilde{M}^1(p)^{-1}=n^{-2}(1\ldots1)C(p)^{-1}\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}.$$

(This procedure is the operation O of [2].)

(v) Take the lowest-order terms of $\tilde{M}^1(p)$ in g and p. After somewhat lengthy algebra we find the following formula:

$$\tilde{M}^1(p)\approx\beta g+\alpha J p^2 \tag{8}$$

which is to be compared with the lowest-order terms of $\tilde{M}(p)$:

$$\tilde{M}(p) \approx g + Jp^2$$

(see (6)). Equation (8) corresponds to the 'self-similarity equation' (3).

β and α in (8) are given by:

$$\beta = n = \text{number of sites in a block} \tag{9}$$

$$\alpha = f + \det(D)/\det(B) \tag{10}$$

where f is the number of sites in a block that has interactions with sites in the adjacent block in the positive direction of the first axis. D is a $n \times n$ matrix and B is a $(n-1)\times(n-1)$ matrix. Both are determined from the structure of a block. B is obtained from a $n \times n$ matrix $C = C(p=0)$ in (7) by taking the left-top part of C with $n-1$ rows and columns. D is obtained using B:

$$D = \begin{bmatrix} B & E \\ {}^{t}E & 0 \end{bmatrix}$$

where E is a $n-1$ component column vector defined by:

$E_j = 1$, if the site j is connected to a site in the adjacent block in the positive direction of the first axis;

-1, if j is connected to a site in the adjacent block in the negative direction of the first axis;

0 otherwise.

The matrix C can be obtained by the following rule. First make a 3×3 torus out of a block in a natural way; i.e. impose periodic boundary conditions at the boundary of the block. Then, for $j \neq k$:

$$\begin{aligned} C_{jk} &= -1 \quad \text{if the sites } j \text{ and } k \text{ are connected} \\ &\qquad\quad \text{(nearest-neighbour pair) on the torus,} \\ &= 0 \qquad \text{otherwise;} \end{aligned}$$

and

$$C_{jj} = 0.$$

In obtaining these formulae, we have made use of the fact that we are considering the fractals that are symmetric with respect to the interchange in the coordinate axis. Using the scaling relation (4) with β and α as given by (9) and (10), we obtain $\tilde{d}(\text{POP})$:

$$\tilde{d}(\text{POP}) = \frac{2 \log \beta}{\log(\beta/\alpha)}. \tag{11}$$

This completes the algorithm of the calculation of the POP approximation.

4. Results: the approximate values of the spectral dimensions

We list the results of our calculations. First, we note that for a regular lattice, the calculation can be done 'analytically'. The direct application of (9)–(11) to a d-dimensional (hyper-)cubic lattice gives the exact answer, as expected:

$$\tilde{d}(\text{POP; regular lattice}) = \tilde{d}(\text{regular lattice}) = d.$$

This result serves as a check of (9)–(11). See the appendix for the calculation.

There seem to be no clever methods to evaluate $\tilde{d}$(POP) for the Sierpinski-type carpets: we have to calculate the determinants of matrices in (10) by force. For example, in the case of the (original) Sierpinski carpet, this means to calculate the determinants of a 8×8 matrix and a 7×7 matrix. In [2] we performed these calculations by hand, and found it rather time consuming. Fortunately, computer calculations save our time. The result of the calculation by hand in [2] is used to check our computer programs.

(i) SC3* (three-dimensional dual Sierpinski carpets)

$$\tilde{d}(\text{POP}; \text{SC3a}^*) = 2\log 26/\log(884/93) \approx 2.89$$

$$\alpha = 93/34 \qquad \beta = 26$$

$$\tilde{d}(\text{POP}; \text{SC3b}^*) = 2\log 20/\log(40/3) \approx 2.31$$

$$\alpha = 3/2 \qquad \beta = 20.$$

The exact values of $\tilde{d}$ for the Sierpinski carpet and its generalisations are not known.

(ii) SC2* (two-dimensional symmetric dual Sierpinski-type carpets).

As we mentioned in § 2.2, we will consider various kinds of two-dimensional symmetric Sierpinski-type carpets by taking away various combinations of squares to form a block. We will restrict ourselves to the cases where the resulting network becomes totally connected. There are fourteen networks that satisfy our criterion. We classify them by the number of squares that we take away.

(i) SC2 with one hole in a block; SC2$\{h\}$ with $h = 3$, 5, or 7.

$$\tilde{d}(\text{POP}; \text{SC2}\{3\}^*) = \tilde{d}(\text{POP}; \text{SC2}\{5\}^*) = \tilde{d}(\text{POP}; \text{SC2}\{7\}^*)$$

$$= 2\log 8/\log(56/5) \approx 1.72$$

$$\alpha = \tfrac{5}{7} \qquad \beta = 8.$$

$\tilde{d}$(POP; SC$\{5\}$) has been given in the previous work [2].

(ii) SC2 with two holes; SC2$\{h\}$ with $h = 35, 57, 37$ or 19. They have a common value:

$$\tilde{d}(\text{POP}; \text{SC2}\{\text{two holes}\}^*) = 2\log 7/\log(84/5) \approx 1.38$$

$$\alpha = \tfrac{5}{12} \qquad \beta = 7.$$

(iii) SC2 with three holes; SC2$\{h\}$ with $h = 139, 179, 236$ or 478. They have a common value:

$$\tilde{d}(\text{POP}; \text{SC2}\{\text{three holes}\}^*) = 2\log 6/\log(180/11) \approx 1.28$$

$$\alpha = \tfrac{11}{30} \qquad \beta = 6.$$

(iv) SC2 with four holes; SC2$\{h\}$ with $h = 1379$, 2356 or 4578. They have a common value:

$$\tilde{d}(\text{POP}; \text{SC2}\{\text{four holes}\}^*) = 2\log 5/\log 15 \approx 1.19$$

$$\alpha = \tfrac{1}{3} \qquad \beta = 5.$$

Following observations are made from our results for the two-dimensional symmetric Sierpinski-type carpets, $\tilde{d}$(POP; SC2$\{h\}^*$).

(i) They satisfy

$$1 < \tilde{d}(\text{POP}; \text{SC2}\{h\}^*) < 2$$

irrespective of the shape of the generator (block). In particular, they are non-integer.

(ii) They depend on the number of holes (squares that have been taken away), but do not depend on the details of which square has been taken away.

(iii) Regarded as a function of the number of holes, $\tilde{d}$(POP; SC2$\{h\}^*$) is a decreasing function.

In [2], we reported our results for d-dimensional Sierpinski gaskets, $\tilde{d}$(POP; SGd), which we compared to the known exact values $\tilde{d}$(SGd) [7, 8]. For completeness, we reproduce the results here:

$$\tilde{d}(\text{POP; SG}d) = 2\log(d+1)/\log(2d+2)$$

$$\beta = d+1 \qquad \alpha = 1/2.$$

The exact values are:

$$\tilde{d}(\text{SG}d) = 2\log(d+1)/\log(d+3).$$

We make some observations on the properties of $\tilde{d}$(POP; SGd).

(i) For all d, they are smaller than the exact values.

(ii) They satisfy:

$$1 < \tilde{d}(\text{POP; SG}d) < 2$$

for all d. This property is shared with the exact values.

(iii) Regarded as a function of d, $\tilde{d}$(POP; SGd) is an increasing function. This property is also shared with the exact values.

(iv) $\tilde{d}$(POP; SGd) and $\tilde{d}$(SGd) share the same $d \to \infty$ limit, $\tilde{d} \to 2$.

5. Discussions

Based on the observations made in the previous section, we conjecture the following on the nature of POP approximations.

(i) The integer part of the value is correct.

(ii) It supplies a lower bound to the correct value.

(iii) It becomes more accurate as the fractal network becomes closer to regular lattices. It is not so good for fractals which are generated by a block with relatively many bonds taken away. In particular, we expect better accuracy for the (original) Sierpinski carpet than for the Sierpinski gasket, and we also expect better accuracy for SC3a than for SC2. It becomes exact for regular lattices.

(iv) The order of the values is correctly given, at least between the fractals of similar type. For example,

$$\tilde{d}(\text{POP; SC2}\{5\}) > \tilde{d}(\text{POP; SC2}\{19\}).$$

From these conjectures and from the results for SC3, we expect that $\tilde{d}$(SC3a*) and $\tilde{d}$(SC3b*), assuming their existence, will take some non-integer values between 2 and 3. We conjecture that $\tilde{d}$(SC3a*) is slightly larger than 2.89 but close to this value.

Finally, we would like to make some comments on some of the remaining problems concerning the POP method.

We have seen that the POP method gives non-integer values of spectral dimensions. It is interesting to know why it is possible to obtain a non-integer answer by using the Fourier transformation method, which should give exact results only for integer dimensionalities. It may have something in common with perturbative renormalisation group methods.

In the present work, we have restricted ourselves to those fractals that are symmetric with respect to the interchange of the coordinate axis. We can, however, use the POP method even if the network is asymmetric. In such a case the effective interactions in different directions, say J_1 and J_2, may be different. Calculations become somewhat involved because we have to calculate the ratio J_1/J_2. This is a non-linear problem, and an iterative method has to be introduced: we must trace the renormalisation group flow in (J_1, J_2) space to find the fixed point. It may be interesting to consider such cases, because we have the problem of 'bifurcation of solutions' for the asymmetric cases; it is the problem that some of the components of J might vanish, even if one starts with non-zero values. These phenomena have been proved to exist for asymmetric generalisations of the Sierpinski gasket [5].

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Appendix

In this appendix, we show that the POP approximation gives the exact value of $\tilde{d}$ for d-dimensional (hyper-)cubic lattices.

Let us perform the POP approximation with a block size L^d. From equation (9):

$$\beta = n = L^d.$$

To compute α by (10), we need a constant f, a matrix C, and a vector E. Note that the elements of C and E are indexed by coordinates $x = (x_1, \ldots, x_d)$ of sites in the block

$f = L^{d-1}$

$C_{xy} = 2d$

$C_{xy} = -1$ if $x \neq y$ and x and y form a nearest-neighbour pair, assuming periodic BC at the boundary of the block

$C_{xy} = 0$ otherwise.

E is the top $n-1$ rows of a n-component column vector F, where F is given by

$F_x = 1$ if $x_1 = L$

$F_x = -1$ if $x_1 = 1$

$F_x = 0$ otherwise.

We introduce a matrix U (also indexed by the coordinates of sites in the block), which we define by:

$$U_{x\xi} = \frac{1}{\sqrt{n}} \exp(2\pi i x\xi / L).$$

This matrix satisfies:

$$UU^{+} = U^{+}U = 1$$

$$(U^{+}CU)_{\xi\eta} = \delta_{\xi,\eta}\left(4 \sum_{\mu=1}^{d} \sin^2(\pi\xi_\mu / L)\right)$$

$$(U^{+}F)_{\xi} = \frac{1}{\sqrt{n}} \frac{n}{L}[1 - \exp(-2\pi i \xi_1 / L)]\delta_{\xi_2,L} \ldots \delta_{\xi_d,L}.$$

Using these equations and elementary properties of matrix determinants, we can evaluate α. The result is:

$$\alpha = L^{d-2}.$$

From equation (10), we arrive at

$$\tilde{d}(\text{POP; regular lattice}) = d.$$

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